Controller Design via Specifier/Implementer Interface: General Idea and Two Examples

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Standard control/system design

• control/system designer designs reference controller that works (achieves required closed-loop performance)

• hardware/software/network implementer implements a controller that is close to the reference controller, except for, e.g.,
  – quantizer errors
  – fixed-point or other roundoff/overflow errors
  – coefficient truncation errors
  – timing errors

• can lead to
  – system that doesn’t work, or
  – over-design in implementation
Specifier/implementer interface

• specifier (control/system designer) and implementer (hardware/software/network implementer) interact through an interface

• specifier provides
  – reference design, and
  – performance certificate

• system designer warrants that overall system will work, if implemented controller passes certification

• implementer uses any method, architecture, etc., provided controller implementation passes certification
  – implemented controller can be far from reference design
Control/system designer’s role

- decide on set of acceptable performance, \( i.e., \) (closed-loop) specifications

- find a set of tests that certify closed-loop performance (using tools of robust control, \( e.g., \) LMIs, . . . )

- reference controller shows that specs can be achieved

- simple, typical case
  - reference controller minimizes some scalar performance measure
  - set of acceptable designs are those within 10\% (say) of optimality
  - use Bellman (optimal value) function as Lyapunov function for certificate
Implementer’s role

• implement simple (cheap, ...) controller that passes certification

• can use reference controller as (feasible) starting point
(Some) publications


Controller coefficient truncation using Lyapunov performance certificate
System

• LTI plant with process noise $w$, sensor noise $v$

$$x_p(t + 1) = A_p x_p(t) + B_p u(t) + w(t), \quad y(t) = C_p x_p(t) + v(t)$$

• LTI controller

$$x_c(t + 1) = A_c x_c(t) + B_c y(t), \quad u(t) = C_c x_c(t) + D_c y(t)$$

• $A_c$, $B_c$, $C_c$ and $D_c$ depend on design parameters or controller coefficients $\theta \in \mathbb{R}^N$ (typically entries of $A_c$, $B_c$, $C_c$ and $D_c$)

• must choose $\theta \in \mathcal{C}$, set of acceptable controller designs

• we are given reference controller design $\theta^{\text{nom}} \in \mathcal{C}$
Controller coefficient truncation problem (CCTP)

- find lowest complexity controller among acceptable designs

$$\text{minimize } \sum_{i=1}^{N} \phi_i(\theta_i)$$
subject to $\theta \in C$

- $\phi_i(\theta_i)$ gives complexity of $\theta_i$, e.g., number of bits needed to express fractional part of binary expansion of $\theta_i$, CSD complexity of $\theta_i$, . . .

- in general case, CCTP is difficult (combinatorial) problem
  - global optimization techniques (e.g., branch-and-bound) can only handle small instances
  - in any case, global solution not needed
The algorithm

- initialize algorithm with nominal design

- at each step
  - choose index $i$ randomly and fix all parameters except $\theta_i$
  - subroutine `interv` finds interval $[l, u]$ of acceptable values for $\theta_i$
  - subroutine `trunc` finds a value of $\theta_i$ in $[l, u]$ with lower complexity

- repeat until there is no change in $\theta$

- run algorithm several times; final choice is best controller coefficient vector found
Lyapunov performance certificate

• use Lyapunov performance certificate:

\[ C = \{ \theta \mid \exists \nu \ L(\theta, \nu) \succeq 0 \} \]

• \( L \) is a bi-affine symmetric matrix function of \( \nu, \theta \)

\[ L(\theta, \nu) = L_0 + \sum_{i=1}^{N} \theta_i L_i \]

• can be done for a huge variety of control system performance specifications
Interval computation

• find value of $\nu$ that (say) maximizes $\log \det L(\theta, \nu)$
  – an easily solved convex problem

• fix $\nu$ and take

$$l = \inf \{ z \mid L((\theta_1, \ldots, \theta_{i+1}, z, \theta_{i+1}, \ldots, \theta_N), \nu) \succeq 0 \},$$
$$u = \sup \{ z \mid L((\theta_1, \ldots, \theta_{i+1}, z, \theta_{i+1}, \ldots, \theta_N), \nu) \succeq 0 \}.$$ 

– $l$, $u$ can be found via generalized eigenvalues
State feedback controller with LQR cost specification

- plant: \( x(t + 1) = Ax(t) + Bu(t), \quad x(0) \sim \mathcal{N}(0, \Sigma) \)

- linear state feedback controller: \( u(t) = Kx(t); \ (\theta \text{ gives entries of } K) \)

- performance measure is LQR cost

\[
J(K) = \mathbb{E} \left[ \sum_{t=0}^{\infty} x(t)^T Q x(t) + u(t)^T R u(t) \right] = \text{Tr}(\Sigma P)
\]

where \( P \) is solution of Lyapunov equation

\[
(A + BK)^T P (A + BK) - P + Q + K^T R K = 0
\]

- \( K^{\text{nom}} \) is optimal state feedback controller (minimizes \( J \))
• acceptable controller designs are those that are $\epsilon$-suboptimal:

$$C = \{K \mid J(K) \leq (1 + \epsilon)J^{\text{nom}}\}$$

• Lyapunov performance certificate:

$$P - (A + BK)^T P (A + BK) \succeq Q + K^T R K$$

$$\text{Tr}(\Sigma P) \leq (1 + \epsilon)J^{\text{nom}}$$

$$P \succeq 0$$

• can be expressed as LMIs, bi-affine in $P (\nu)$ and $K (\theta)$
Example

• $A \in \mathbb{R}^{10 \times 10}, B \in \mathbb{R}^{10 \times 5}$ randomly generated

• $\Sigma = I, Q = I, R = I$

• $\Phi$ is total number of bits in fractional part of coefficients

• fractional part of each entry of $K^{\text{nom}}$ expressed with 40 bits; $\Phi(K^{\text{nom}}) = 2000$ bits.

• $\epsilon = 15\%, \ i.e.,$ acceptable feedback controllers are up to $15\%$-suboptimal
Typical run

converges to a complexity of 85 bits in 50 iterations
Final design

best design after 100 random runs achieves $\Phi(K) = 75$ bits
(1.5 bits per coefficient)
Initial and final designs

- initial design

\[ K^{\text{nom}} = \begin{bmatrix}
-0.48 & -0.17 & -0.18 & 0.13 & 0.09 & 0.04 & 0.29 & 0.20 & 0.03 & -0.06 \\
0.11 & 0.10 & -0.01 & -0.09 & 0.01 & -0.02 & -0.07 & 0.07 & 0.33 & 0.07 \\
0.39 & -0.03 & 0.16 & -0.03 & -0.28 & -0.05 & -0.22 & -0.05 & -0.08 & 0.23 \\
-0.66 & 0.10 & -0.13 & 0.10 & -0.01 & 0.35 & -0.05 & -0.14 & 0.02 & -0.07 \\
-0.21 & 0.11 & 0.20 & 0.10 & -0.26 & 0.06 & 0.04 & -0.02 & 0.13 & -0.21 \\
\end{bmatrix} \\
\]

- final design

\[ K^{\text{trunc}} = \begin{bmatrix}
-1/2 & -1/8 & -1/8 & 1/8 & 1/8 & 1/4 & 1/4 \\
1/8 & 1/8 & -1/8 & 3/8 \\
\end{bmatrix} \\
\]

(Entries not shown are zero)
Synthesis of state-feedback controllers for systems with timing jitter
Continuous-time system model

• continuous-time LTI plant

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \]

• controller sample times \(0 = t_0 < t_1 < t_2 < \ldots, \text{i.e.,} \)

\[ u(t) = u_i, \quad t_i \leq t < t_{i+1}, \quad i = 0, 1, \ldots \]
Discrete-time model

• with $x_i = x(t_i)$, we have time-varying linear system

$$x_{i+1} = A^d(s_i)x_i + B^d(s_i)u_i$$

• $s_i = t_{i+1} - t_i$ is $i$th intersample time

• $A^d(s) = e^{sA}$, $B^d(s) = Z(s)B$, with

$$Z(s) = \int_0^s e^{\tau A} d\tau$$
Timing model

- perfect clock, sample period $T$ corresponds to $t_i = iT$

- gives (time-invariant) *nominal closed-loop system*

$$x_{i+1} = A_{\text{nom}}^d x_i + B_{\text{nom}}^d u_i$$

where $A_{\text{nom}}^d = A^d(T)$ and $B_{\text{nom}}^d = B^d(T)$

- our model: $t_{i+1} - t_i$ is near, but not exactly equal to, $T$

$$T - \Delta \leq t_{i+1} - t_i \leq T + \Delta, \quad i = 0, 1, \ldots$$

$\Delta$ is maximum possible jitter
Performance measure

- use continuous-time LQR cost:

\[
J(x_0, u) = \int_0^\infty \left( x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) d\tau
\]

\[
= \sum_{i=0}^{\infty} \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} x_i \\ u_i \end{bmatrix}
\]

where

\[
\Gamma(s) = \begin{bmatrix}
Q^d(s) & S^d(s) \\
S^d(s)^T & R^d(s)
\end{bmatrix}
\]

\[
= \int_0^s \begin{bmatrix}
e^{\tau A} & Z(\tau) \\
0 & I
\end{bmatrix}^T \begin{bmatrix} Q & 0 \\
0 & R
\end{bmatrix} \begin{bmatrix} e^{\tau A} & Z(\tau) \\
0 & I
\end{bmatrix} d\tau
\]
\begin{itemize}
  \item $Q^d(s)$, $S^d(s)$, and $R^d(s)$ can be computed numerically or analytically.
  \item $\Gamma(s) \succeq 0$.
  \item $J(x_0, u)$ is a convex functional in $x_0$ and the sequence $\{u_i\}$.
\end{itemize}
Linear state-feedback controller

- linear state-feedback controller

\[ u_i = K x_i, \quad i = 0, 1, \ldots \]

where \( K \in \mathbb{R}^{m \times n} \) is state feedback gain

- discrete-time, linear time-varying system is

\[ x_{i+1} = (A^d(s_i) + B^d(s_i)K)x_i \]

- cost becomes

\[ J(x_0) = \sum_{i=0}^{\infty} x_i^T \begin{bmatrix} I & K \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} I & K \end{bmatrix} x_i \]
• for a fixed sample time sequence, the cost $J(x_0)$ is a convex quadratic function in the initial state $x_0$

• define the **worst-case cost** as

$$J_{wc}(x_0) = \sup \{ J(x_0) \mid \{ t_i \} \text{ satisfies timing jitter model} \}$$

• $J_{wc}$ is convex and homogeneous of degree 2
Worst-case relative performance degradation

- cost for nominal system is

\[ J_{\text{nom}}(x_0) = x_0^T P_{\text{nom}} x_0 \]

where \( P_{\text{nom}} \) is solution of Lyapunov equation

\[
\left( A_{\text{nom}}^d + B_{\text{nom}}^d K \right)^T P \left( A_{\text{nom}}^d + B_{\text{nom}}^d K \right) - P \\
+ Q^d(T) + S^d(T) K + K^T S^d(T)^T + K^T R^d(T) K = 0
\]

- worst-case relative performance degradation

\[
\eta = \sup_{x_0 \neq 0} \frac{J_{\text{wc}}(x_0) - J_{\text{nom}}(x_0)}{J_{\text{nom}}(x_0)}
\]
Upper bound via quadratic Lyapunov function

• suppose $V(z) = z^T P z$ satisfies $V(z) \geq J(z)$ for all $z$, and all $\{t_i\}$

• then $V(z) \geq J_{wc}(z)$ for all $z$, and $\eta \leq \lambda_{\text{max}}(P, P_{\text{nom}}) - 1$

• sufficient condition: $P \succeq 0$, and

$$
\begin{bmatrix}
I \\
K
\end{bmatrix}^T (F(s) + \Gamma(s)) \begin{bmatrix}
I \\
K
\end{bmatrix} \preceq 0 \quad \text{for } T - \Delta \leq s \leq T + \Delta,
$$

where

$$F(s) = \begin{bmatrix}
A^d(s)^T P A^d(s) - P & A^d(s)^T P B^d(s) \\
B^d(s)^T P A^d(s) & B^d(s)^T P B^d(s)
\end{bmatrix}$$

• second inequality is a semi-infinite LMI in the matrix $P$
Best upper bound

- choose $P$ to get smallest upper bound on $\eta$ by solving

$$\begin{align*}
\text{minimize} & \quad \lambda_{\text{max}}(P, P_{\text{nom}}) - 1 \\
\text{subject to} & \quad \begin{bmatrix} I \\ K \end{bmatrix}^T (F(s) + \Gamma(s)) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0 \quad \text{for } T - \Delta \leq s \leq T + \Delta, \\
& \quad P \succeq 0.
\end{align*}$$

with variable $P$

- a quasiconvex problem, easily solved

- discretize semi-infinite constraint to get standard problem:

$$s_j = T + (2j - N)\Delta/N, \quad j = 0, 1, \ldots, N$$
Lower bound

heuristic method for finding lower bound on $J_{wc}$:

- generate initial state $x_0$

- choose timing sequence to greedily maximize increase of Lyapunov function $V$
State-feedback controller synthesis

• (discretized) problem:

\[
\text{minimize} \quad \lambda_{\text{max}}(P, P_{\text{nom}}) - 1 \\
\text{subject to} \quad \begin{bmatrix} I \\ K \end{bmatrix}^T (F_i + \Gamma_i) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0, \quad j = 0, 1, \ldots N \\
P \succeq 0
\]

with optimization variables \( P \) and \( K \)

• not convex as stated, but change of variables yields quasiconvex problem
Change of variables

• let $Y = P^{-1}$ and $V = KP^{-1}$

• problem becomes

$$\text{maximize} \quad \lambda_{\min}(Y, P_{\text{nom}}^{-1})$$

subject to

$$\begin{bmatrix} Y & (A_i^dY + B_i^dV)^T \\ A_i^dY + B_i^dV & Y \end{bmatrix} \succeq \begin{bmatrix} W_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, 1, \ldots,$$

$$\begin{bmatrix} W_i & Y & V^T \\ Y & V & \Gamma_i^{-1} \end{bmatrix} \succeq 0, \quad i = 0, 1, \ldots, N$$

with optimization variables $Y$, $V$, and $W_i$, for $i = 0, 1, \ldots, N$

• from solutions $Y^*$, $Z^*$ we recover

$$P^* = (Y^*)^{-1}, \quad K^* = V^*(Y^*)^{-1}$$
Example

• \( n = 10, \ m = 2, \) with IID \( N(0, 1) \) entries

• \( Q = I, \ R = I \)

• \( T = 0.1 \)

• \( N = 5 \)
• lower and upper bounds on $\eta$ with $K = K_{lqr}$ (blue) and $K = K^*$ (red)
• lower and upper bounds are indistinguishable, i.e., $\eta$ determined to within very small interval